The University of Chicago
FOUNDED BY JOHN D. ROCKEFELLER

ON THE CONVERGENCY OF THE SERIES USED IN THE DETERMINATION OF THE ELEMENTS OF PARABOLIC ORBITS, AND THE ERRORS INTRODUCED IN THE ELEMENTS BY IMPERFECTIONS OF THE OBSERVATIONS

A DISSERTATION

SUBMITTED TO THE FACULTIES OF THE GRADUATE SCHOOLS OF ARTS, LITERATURE, AND SCIENCE, IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

(DEPARTMENT OF ASTRONOMY)

BY
WILLIAM ALBERT HAMILTON

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PART I.

1. *Introductory.*—The elements of the orbit of a comet are usually determined by means of the data obtained at three separate, complete observations; and it often becomes a question of importance to the astronomer as to the suitability, or perhaps we might say the sufficiency, of such a set of observations to determine with accuracy the required elements. He is confronted, on the one hand, with a set of formulae somewhat complex in their nature, and which are subject to limitations in their application owing to the properties of the various functions involved in their construction; on the other hand, the data of observation are subject to limitations owing to unavoidable inaccuracies in the construction of the telescope and the multitude of items which fall under the class known as errors. It is thus both a mathematical and a physical problem with which he has to deal, and it becomes important, first, that a careful analysis be made of the properties of the formulas and the conditions under which they may be applied, and, secondly, that the errors which present themselves in the observations, in spite of the greatest care and skill on the observer's part, may not be allowed to become obscured in the final results of the computation. It has been the purpose of the study of which this paper is a partial result to investigate the formulas for computing cometary orbits from each of these two standpoints. In pursuance of this plan, we have investigated, among other questions, the nature of the functions usually known as the "ratios of the triangles;" and have found the precise conditions under which they may be developed into converging power series of the time intervals between the observations.[1] This discussion is given in the first part of this paper. In another part of the investigation we have found the effects of the errors of the observations upon the computed elements of the orbit of a comet, using Obers's method as a basis of the study. To this is added the results of a computation, by use of the formulæ so deduced, of the differentials of error in an actual case. We proceed to discuss the ratios

[1] That the series converge for sufficiently small values of the time-intervals follows from CAUCHY'S existence theorems published in 1842 (*Coll. Works*, 1st series, Vol. VII, pp. 5 f.). PROFESSOR PAUL HARZER has attained the same general results in the *Publications of the Kiel Observatory*, Vol. XI, Part 2, by direct treatment of the series. Evidently the results of both Cauchy and Harzer are of the nature of existence theorems and were not intended for practical use by computers, because the true radius of convergence was not found.

of the triangles and convergency of the series. We use the following notation:

2. *Notation.*—Let t_1, t_2, t_3 denote the first, second, and third times of observation, respectively. And if k denote the Gaussian constant and m the mass of the comet in terms of the mass of the sun taken as unity, then the differential equations of motion of the comet referred to the sun's center as origin of co-ordinates are,

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{k^2(1+m)x}{r^3} \\ \frac{d^2y}{dt^2} &= -\frac{k^2(1+m)y}{r^3} \\ \frac{d^2z}{dt^2} &= -\frac{k^2(1+m)z}{r^3} \end{aligned} \tag{1}$$

where r is the heliocentric distance of the comet, and x, y, z are its rectangular cartesian co-ordinates. In all practical cases m will be infinitesimal in comparison with the mass of the sun, and therefore may be neglected. Furthermore, if we so change the unit of time that the new unit shall be equal to the old when the latter has been multiplied by k, and denote the time when expressed in the new units by $t-t_0$, where t_0 is any particular epoch, we may express these equations of motion very simply, thus:

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{x}{r^3} \\ \frac{d^2y}{dt^2} &= -\frac{y}{r^3} \\ \frac{d^2z}{dt^2} &= -\frac{z}{r^3} \end{aligned} \tag{2}$$

In these equations the attractions of all the bodies of the solar system are neglected, except that of the sun.

3. *Preliminary notions.*—Suppose now the co-ordinates of the comet at the time t_0 to be x_0, y_0, z_0, and its velocities to be $\frac{dx_0}{dt}$, $\frac{dy_0}{dt}$, $\frac{dz_0}{dt}$; then, at any other time, the co-ordinates and velocities are functions of these initial conditions and $t-t_0$; or, as we say,

$$x = f\left(x_0,\ y_0,\ z_0;\ \ \frac{dx_0}{dt},\ \frac{dy_0}{dt},\ \frac{dz_0}{dt};\ \ t-t_0\right);$$

with similar expressions for the other co-ordinates and the velocities.

Now it is known from the theory of differential equations that the co-ordinates and velocities are expansible into power series in $(t-t_0)$ of the form

$$x=f\left(x_0 \ldots \frac{dx_0}{dt};\quad 0\right)+\left[\frac{\partial f}{\partial t}\right]_0(t-t_0)+\left[\frac{\partial^2 f}{\partial t^2}\right]_0\frac{(t-t_0)^2}{\underline{|2}}+\cdots \tag{3}$$

which have finite radii of convergency, if r does not vanish for $t-t_0=0$.[1] In the partial derivatives above $t-t_0$ is to be placed equal to zero after differentiation. Hence, as seen from (3),

$$\left[\frac{\partial f}{\partial t}\right]_0=\frac{dx_0}{dt}\,,\ \ldots,\ \left[\frac{\partial^n f}{\partial t^n}\right]_0=\frac{d^n x_0}{dt^n}\,,\ \ldots. \tag{4}$$

From equation (2) we obtain

$$\begin{aligned}\frac{d^2x_0}{dt^2}&=-\frac{x_0}{r_0^3}\,,\\ \frac{d^3x_0}{dt^3}&=\frac{3x_0}{r_0^4}\frac{dr_0}{dt}-\frac{1}{r_0^3}\frac{dx_0}{dt}\,,\ \ldots.\end{aligned} \tag{5}$$

Equations (5) enable us to find the coefficients for the developments of the type (3), by means of which the co-ordinates and velocities of the comet at any time t are expressed as power series of the time intervals $t-t_0$, and coefficients depending only upon the co-ordinates and velocities at the initial time t_0. By means of these developments of the co-ordinates the so-called ratios of the triangles are built up in the form of series which depend upon particular time intervals selected from those determined by the three observations. It is in regard to these latter series that we wish to find the conditions of convergency; and it is at once evident that their convergency will depend upon the convergency of the series of the type (3), since the ratios of the triangles are functions of the co-ordinates alone.

4. *Convergency of series.*—From well-known theorems of the theory of functions of complex variables it is clear that any expansions whatever of the ratios of the triangles into power series for given time intervals and initial conditions cannot have greater radii of convergency than the values which are determined by the poles and branch points of the expressions of those ratios as functions of the time intervals. First, however, we study the nature of the functions which express x, y, z in terms of t, and from these find the true radius of convergency.

[1] JORDAN, *Cours d'analyse*, Vol. III, p. 99.

5. *Co-ordinates as functions of the time.*—From the geometrical relations of the orbit of the comet we have the relations

$$
\begin{aligned}
x &= r[\cos(v+\omega)\cos\Omega - \sin(v+\omega)\sin\Omega\cos i] , \\
y &= r[\cos(v+\omega)\sin\Omega + \sin(v+\omega)\cos\Omega\cos i] , \\
z &= r[\sin(v+\omega)\sin i] ,
\end{aligned} \tag{6}
$$

where v is the true anomaly, ω is the argument of latitude of the perihelion, Ω is the longitude of the node, and i is the inclination of the orbit to the ecliptic. The last three quantities are independent of the time; while v and r are expressible in terms of t by means of the relations

$$
\begin{aligned}
r &= \frac{p}{1+\cos v} , \\
\tan\frac{v}{2} + \frac{1}{3}\tan^3\frac{v}{2} &= \frac{2}{p^{\frac{3}{2}}}(t-\Pi) ,
\end{aligned} \tag{7}
$$

where p is the latus rectum of the parabolic orbit of the comet and Π is the time of perihelion passage. Π and t are thought of as expressed in units described in section 2 above—a usage which we shall continue throughout this paper.

6. *The solution of the cubic.*—By means of the equation (7) we are enabled to express x, y, and z in terms of the time intervals $t-\Pi$. In order to do this we introduce the auxiliaries

$$
\begin{aligned}
\tau &= \frac{3k(t-\Pi)}{p^{\frac{3}{2}}} , \\
\phi &= \tan\frac{v}{2} .
\end{aligned} \tag{8}
$$

Then the last equation of (7) becomes

$$
\phi^3 + 3\phi - 2\tau = 0 . \tag{9}
$$

This is the so-called normal form of the cubic in the quantity ϕ. Its solutions by Cardan's formula are

$$
\begin{aligned}
\phi_1 &= q_1 + q_2 , \\
\phi_2 &= \epsilon q_1 + \epsilon^2 q_2 , \\
\phi_3 &= \epsilon^2 q_1 + \epsilon q_2 ,
\end{aligned} \tag{10}
$$

where $q_1 = (\tau + \sqrt{1+\tau^2})^{\frac{1}{3}}$, $q_2 = (\tau - \sqrt{1+\tau^2})^{\frac{1}{3}}$, and 1, ϵ, ϵ^2 are cube roots of unity.[1]

[1] See Burnside and Panton, *Theory of Equations*, p. 108.

7. *Branch points.*—We wish to express ϕ in a power series in t, and must, therefore, find the branch points and poles of the function. At once we have the branch points $\tau = i$ and $\tau = -i$, where $i = \sqrt{-1}$. Also $\tau = \infty$ is a branch point, as is easily seen by putting $\tau = \frac{1}{\tau'}$, and letting τ' approach zero. This is the same as putting $\tau = \infty$, and we easily find that all three solutions have the same value at this point. If now we consider a Riemann surface of three sheets with branch points at $\tau = i$, $\tau = -i$, $\tau = \infty$, then by the theory of functions of a complex variable we know that the quantity ϕ is a uniform function of position on this surface.

8. *Connection of the sheets.*—In order to get a clear idea of the surface, it is necessary to find what sheets pass into each other at the two branch points which are in the finite part of the plane. To do this we need to follow only the purely imaginary values of τ; for the two branch points in question are on the axis of pure imaginaries. Indeed, we may also consider the branch point $\tau = \infty$ to be on this same axis.

In order to simplify matters and at the same time render the reasoning clearer we make the transformation

$$\tau = i \cos\theta\ , \tag{11}$$

where θ is real or complex. Then q_1 and q_2 become

$$q_1 = [i(\cos\theta - i\sin\theta)]^{\frac{1}{3}} = -ie^{-\frac{i\theta}{3}},$$

$$q_2 = [i(\cos\theta + i\sin\theta)]^{\frac{1}{3}} = -ie^{\frac{i\theta}{3}}.$$

And since we may write $\epsilon = e^{\frac{2\pi i}{3}}$, $\epsilon^2 = e^{-\frac{2\pi i}{3}}$, we obtain from (10)

$$\begin{aligned}
\phi_1 &= -i\left(e^{\frac{i\theta}{3}} + e^{-\frac{i\theta}{3}}\right) = -2i\cos\frac{\theta}{3}, \\
\phi_2 &= -i\left(e^{\frac{i}{3}(\theta-2\pi)} + e^{-\frac{i}{3}(\theta-2\pi)}\right) = -2i\cos\frac{\theta-2\pi}{3}, \\
\phi_3 &= -i\left(e^{\frac{i}{3}(\theta+2\pi)} + e^{-\frac{i}{3}(\theta+2\pi)}\right) = -2i\cos\frac{\theta+2\pi}{3}.
\end{aligned} \tag{12}$$

Now from (11), if θ takes real values, τ is purely imaginary and takes values between $\tau = i$ and $\tau = -i$; while if θ is a pure imaginary, τ takes pure imaginary values with moduli greater than unity. Only when θ is complex does τ take real or complex values. Hence for our purpose we need consider only imaginary values of θ, or real values of θ, in order to find the connection of the sheets.

We must notice also that τ is a periodic function of θ; hence when we wish τ to trace the line between the two branch points but once, we take the primitive period and consider this alone. Now, in order that τ may take only pure imaginary values while passing from $\tau = -i$ to $\tau = i$, θ must take the real values between o and π, and therefore $\frac{\theta}{3}$ will take the real values between o and $\frac{\pi}{3}$. We get the following correspondence for θ, τ, ϕ_1, ϕ_2, and ϕ_3:

θ	τ	ϕ_1	ϕ_2	ϕ_3
o	i	$-2i$	i	i
$\frac{\pi}{2}$	o	$-i\sqrt{3}$	o	$i\sqrt{3}$
π	$-i$	$-i$	$-i$	$2i$

Denote the branch points $\tau = i$ and $\tau = -i$ by A and A' respectively. Then from the table above we find, according to the period selected, that the two values ϕ_2 and ϕ_3 become equal when τ approaches A; but when τ arrives at A' along the path selected, this does not repeat itself; but instead we have $\phi_1 = \phi_2$. Hence sheets ϕ_2 and ϕ_3 are connected at $\tau = i$; while ϕ_2 and ϕ_1 are connected at $\tau = -i$. It follows that if we start at $\tau = o$ in the τ-surface and make a complete circuit once around A, then o and $i\sqrt{3}$ will change places; while for a circuit around A', o and $-i\sqrt{3}$ will change places. If we draw branch cuts from A to infinity and from A' to negative infinity, the continuation of the sheets when crossing these cuts will be:

$$\text{Along } A \text{ to } \infty, \quad . \quad . \quad . \quad \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 2}.$$

$$\text{Along } A' \text{ to } -\infty, \quad . \quad . \quad . \quad \frac{1 \cdot 2 \cdot 3}{2 \cdot 1 \cdot 3}.$$

All three sheets are connected at $\tau = \infty$. A section along the axis of pure imaginaries will appear as in Fig. 1.

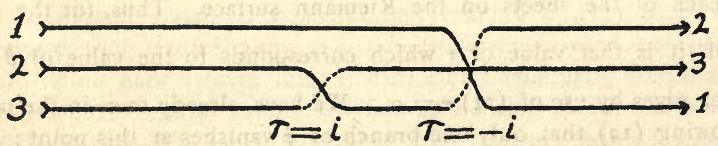

FIG. 1.

It will be of importance for what follows to notice that in the θ-plane the portion which is bounded by the axis of pure imaginaries and the line $\theta = \pi$ is a conform representation of the whole τ-plane, each sheet being represented once in the fundamental region.

9. *Poles of ϕ in the sheets.*—It is well known that where z is a complex variable, the function e^z can become infinite only for infinite values of z. Hence it follows from (12) that ϕ cannot become infinite except for infinite values of θ. Moreover, owing to the periodicity of the function e^z, which makes $e^{z+2\pi i} = e^z$, the above infinite value of θ must be either purely imaginary or perhaps complex with the imaginary part of the complex expression infinitely great. But, from (11), such a value of θ gives τ infinite. Hence it follows that ϕ cannot become infinite except for infinite values of τ. Hence there are no finite poles of ϕ in the sheets of the Riemann surface.

10. *Zeroes in the sheets.*—By use of (12) we are also enabled to find at once the zeroes of ϕ in the τ-surface. The general condition for the vanishing of ϕ is given by either of the conditions

$$\cos\frac{\theta}{3} = 0 \ ,$$

$$\cos\left(\frac{\theta \pm 2\pi}{3}\right) = 0 \ .$$

These are virtually the same, since we may get the one from the other by putting $\theta = \theta \pm 2\pi$. It is then only necessary to find the value of θ for which $\cos\frac{\theta}{3} = 0$. Now, by methods well known to the theory of trigonometry[1] it is readily proved that the only values of θ, real or complex, which satisfy this condition are

$$\theta = \frac{2n + 1}{2}\pi \ ,$$

where n is a positive or negative integer or zero. It follows that only one zero of ϕ is to be found in each fundamental region of the θ-plane for each of the sheets on the Riemann surface. Thus, for the first sheet, it is that value of τ which corresponds to the value of $\theta = \frac{\pi}{2}$ which gives by use of (11) $\tau = 0$. We have already seen in the table following (12) that only one branch of ϕ vanishes at this point; and

[1] See CHRYSTAL, *Algebra*, Vol. II, chap. 29.

that the particular one which vanishes thus is dependent entirely upon the sheet of the Riemann surface in which τ is found.

11. *Résumé.*— We note here the following summary of results as to critical points upon the Riemann surface upon which ϕ is a function of position :

$$\text{Zeroes at} \begin{cases} \tau = 0 \text{ in the } \tau\text{-surface}, \\ \theta = \dfrac{\pi}{2} \text{ in the fundamental region of the } \theta\text{-plane}; \end{cases}$$

$$\text{Poles, none in the finite part of the } \tau\text{-plane}; \tag{13}$$

$$\text{Branch points} \begin{cases} \tau = i , \\ \tau = -i , \\ \tau = \infty . \end{cases}$$

12. *Rational functions of ϕ and τ.*— At this place we state the following theorem which will be useful for later work :

Every rational function of ϕ and τ is a uniform function of position on the same Riemann surface as that which describes ϕ as a function of τ and its branch points are at the same places.[1] It is to be remembered, however, that this theorem does not apply to the *zeroes* and *poles* of such a rational function of ϕ and τ. These may be located otherwise than as described in (13).

13. *Co-ordinates x and y as functions of τ.*— We may write the first two equations of (6) as follows :

$$\begin{aligned} x &= [c \cos v - s \sin v]\left(1 + \tan^2 \frac{v}{2}\right) , \\ y &= [c_1 \cos v - s_1 \sin v]\left(1 + \tan^2 \frac{v}{2}\right) ; \end{aligned} \tag{14}$$

where $p = r(1 + \cos v)$; and

$$\begin{aligned} \frac{2c}{p} &= \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i , \\ \frac{2s}{p} &= \sin \omega \cos \Omega + \cos \omega \sin \Omega \cos i , \\ \frac{2c_1}{p} &= \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i , \\ \frac{2s_1}{p} &= \sin \omega \sin \Omega - \cos \omega \cos \Omega \cos i . \end{aligned} \tag{15}$$

[1] See FORSYTHE, *Theory of Functions*, p. 369.

Using the relations $\cos v = \dfrac{1-\tan^2\frac{v}{2}}{1+\tan^2\frac{v}{2}}$ and $\sin v = \dfrac{2\tan\frac{v}{2}}{1+\tan^2\frac{v}{2}}$, we may write (14), where we put $\tan\frac{v}{2} = \phi$ in the form,

$$\left.\begin{aligned} x &= c - 2s\phi - c\phi^2 \\ y &= c_1 - 2s_1\phi - c_1\phi^2 \end{aligned}\right\}, \tag{16}$$

Now, in the equations (16) c, c_1, s, and s_1 are constants independent of τ, and, by the theorem given in the last article, x and y, considered as functions of τ, are functions of position on the same Riemann surface which defines ϕ as a function of τ, and the branch points of x and y are $\tau = i$, $\tau = -i$, and $\tau = \infty$. Moreover, since c, c_1, s, and s_1 are constants and never infinite, x and y cannot become infinite except where ϕ becomes infinite, viz., at $\tau = \infty$. Hence we have:

Theorem: x and y have poles in the Riemann sheets only at $\tau = \infty$ and they have branch points at $\tau = i$, $\tau = -i$, and $\tau = \infty$.

It follows from the above that x and y are holomorphic functions of τ in the sheets of the Riemann surface except at points $\tau = i$, $\tau = -i$, and $\tau = \infty$. Therefore they may each be expanded into power series with argument $\tau - \tau_0$ in the vicinity of any point $\tau = \tau_0$. These series will be convergent inside of a circle whose center is at τ_0 and whose radius reaches from τ_0 to the nearest of the points $\tau = i$, or $\tau = -i$.

14. *Radius of convergency.*—If in (3) we replace t and t_0 by their corresponding values in τ by relation (8), we have just such an expansion as described in the last article. If we should at the same time take $t_0 = 0$, the expansion in x becomes of the form

$$x = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + \cdots,$$

where a_0, a_1, etc., are constants. This series will be convergent inside a circle whose center is $\tau = 0$, and with radius unity reaching up to the branch points $\tau = i$ and $\tau = -i$. Hence, the true radius of convergency in this case would be $|\tau| = 1$; or from (8)

$$(t - \pi) = \frac{p^{\frac{3}{2}}}{3}. \tag{17}$$

Suppose in (17) we give to p any value, say 1, which would correspond to a perihelion distance of 0.5; then if we make $k = \frac{1}{60}$, which is approximately its value, the case under supposition would give as the

limit for the time interval for which the expansion of x into power series would be convergent, the value 20 days. The same period would hold for the corresponding expansion of y.

If τ_0 were any finite point not equal to zero, say some point on the real axis of the τ-plane, then the radius of the true circle of convergency would be larger than that given above. In this case the radius would be $R_\tau = \sqrt{1 + \tau_0^2}$, which holds for both the x and the y series. The radius of convergency of the corresponding series in t is at once deducible from the series in τ through the relation (8). The relation is always

$$R_t = \frac{p^{\frac{3}{2}}}{3} R_\tau , \tag{18}$$

where the subscripts denote the argument of the series. Since in an absolutely convergent series we are at liberty to change the order of the terms at will, we may express (3) and the corresponding equation in y by use of coefficients of the kind given in (5), as follows :

$$\left.\begin{aligned} x &= Ax_0 + B\frac{dx_0}{dt} , \\ y &= Ay_0 + B\frac{dy_0}{dt} , \end{aligned}\right\} \tag{19}$$

where A and B for their first few terms are

$$\left.\begin{aligned} A &= 1 - \frac{1}{2}\frac{t^2}{r_0^3} + \frac{1}{2}\frac{t^3}{r_0^4}\frac{dr_0}{dt} + \frac{t^4}{24}\left[\frac{1}{r_0^6} - \frac{12}{r_0^5}\left(\frac{dr_0}{dt}\right)^2 + \frac{3}{r_0^4}\frac{d^2r_0}{dt^2}\right] + \cdots , \\ B &= t - \frac{1}{6}\frac{t^3}{r_0^3} + \frac{1}{4}\frac{t^4}{r_0^4}\frac{dr_0}{dt} + \cdots . \end{aligned}\right\} \tag{20}$$

In these series we have taken $t_0 = 0$. They may be written as series in $t - t_0$ by use of the theory of continuation of power series.

Since a power series serves in every way to define the behavior of the function from which it is derived so long as we remain within its circle of convergence, we can deal with the series (19) as with quantities which obey all the laws of ordinary algebra — association, commutation, etc.; such as ordinary polynomials or rational quantities, and the resulting series will be convergent.[1]

15. *Ratios of the triangles.*— We denote the triangle between the positions of two radii vectores of the comet's orbit by the expression $[r_i, r_j]$, where i and j denote the order of any two of the three observations; also in general we denote the co-ordinates of the first, second,

[1] See CHRYSTAL, *Algebra*, Vol. II, pp. 139–43.

and third observations by the subscripts 1, 2, 3 respectively. Now the ratios of the triangles $[r_i, r_j]$ are equal to the ratios of their projections upon any plane, which fact may be expressed thus:

$$\begin{aligned} \frac{[r_2, r_3]}{[r_1, r_2]} &= \frac{x_2 y_3 - y_2 x_3}{x_2 y_1 - y_2 x_1}, \\ \frac{[r_1, r_2]}{[r_1, r_3]} &= \frac{x_2 y_1 - y_2 x_1}{x_1 y_3 - y_1 x_3}. \end{aligned} \tag{21}$$

Let now $x_2, y_2, \frac{dx_2}{dt} \frac{dy_2}{dt}$ be taken as the zero values of the co-ordinates and velocities in the expansions (19) and (20). Then we get

$$\begin{aligned} x_1 &= A_1 x_2 + B_1 \frac{dx_2}{dt}, \\ y_1 &= A_1 y_2 + B_1 \frac{dy_2}{dt}, \\ x_3 &= A_3 x_2 + B_3 \frac{dx_2}{dt}, \\ y_3 &= A_3 y_2 + B_3 \frac{dy_2}{dt}; \end{aligned} \tag{22}$$

where A_1, B_1, A_3, B_3 are defined by:

$$\begin{aligned} A_1 &= 1 - \frac{1}{2}\frac{(t_1 - t_2)^2}{r_2^3} + \dots, \quad B_1 = (t_1 - t_2) - \frac{1}{6}\frac{(t_1 - t_2)^3}{r_2^3} + \dots; \\ A_3 &= 1 - \frac{1}{2}\frac{(t_3 - t_2)^2}{r_2^3} + \dots, \quad B_3 = (t_3 - t_2) - \frac{1}{6}\frac{(t_3 - t_2)^3}{r_2^3} + \dots. \end{aligned} \tag{23}$$

Now, the series (22) are convergent within the same circle. It follows that, since $x_2, y_2, \frac{dx_2}{dt}, \frac{dy_2}{dt}$ are not in general equal to zero, the series (23) are also convergent in this same circle.[1] It follows that since for such series the law of distribution holds[2] the products A_1, B_3 and $A_3 B_1$ are also convergent series. Hence, from the law of addition, we have that $A_1 B_3 - B_1 A_3$ is also convergent.[3]

We are now at liberty to substitute the values of x_1, y_1, x_3, y_3 as given by (22) in the ratios on the right of (21). We get, after the substitution indicated and by canceling the factor $x_2 \frac{dy_2}{dt} - y_2 \frac{dx_2}{dt}$ from the two members of the ratio,

1 *Ibid.*, p. 178, 5. 2 *Ibid.*, pp. 142, 143. 3 *Ibid*, p. 141.

$$\left.\begin{aligned}
\frac{[r_2,\, r_3]}{[r_1,\, r_2]} &= -\frac{B_3}{B_1} = \frac{(t_3 - t_2)}{(t_1 - t_2)}\Big[1 - \frac{1}{6}\frac{(t_3 - t_2)^2 - (t_1 - t_2)^2}{r_2^3} + \\
&\qquad \frac{1}{4}\frac{(t_3 - t_2)^3 + (t_1 - t_2)^3}{r_2^4}\frac{dr_2}{dt}\cdots\Big], \\
\frac{[r_1,\, r_2]}{[r_1,\, r_3]} &= \frac{-B_1}{A_1 B_3 - B_1 A_3} = \frac{(t_1 - t_2)}{(t_3 - t_1)}\Big[1 + \frac{1}{6}\frac{(t_3 - t_1)^2 - (t_2 - t_1)^2}{r_2^3} - \\
&\qquad \frac{1}{4}\frac{(t_3 - t_1)(t_3 - t_2)^2 - (t_3 - t_2)(t_2 - t_1)^2}{r_2^4}\frac{dr_2}{dt}\cdots\Big].
\end{aligned}\right\} \quad (24)$$

where A_1, B_1, A_3, B_3 have the meaning given by (23).

Now, B_1 and B_3 are series with arguments $t_1 - t_2$, and $t_3 - t_2$ respectively. They hold, *i. e.*, are convergent, as long as $t_1 - t_2$ and $t_3 - t_2$ obey the relations

$$\left.\begin{aligned}
|t_1 - t_2| &< \frac{p^{\frac{3}{2}}}{3}\sqrt{\frac{9}{p^3}(t_2 - \Pi)^2 + 1}\,, \\
|t_3 - t_2| &< \frac{p^{\frac{3}{2}}}{3}\sqrt{\frac{9}{p^3}(t_2 - \Pi)^2 + 1}\,.
\end{aligned}\right\} \quad (25)$$

Also the series $A_1 B_3 - B_1 A_3$ which has two arguments, viz., $t_1 - t_2$ and $t_3 - t_2$ is convergent so long as (25) holds.

16. *The zeroes of B_1 and $A_1 B_3 - B_1 A_3$.*—If B_1 should vanish, or if $A_1 B_3 - B_1 A_3$ should vanish, then the fractions on the right of (24) evidently would no longer be legitimate. It is easily seen that the former contingency cannot occur unless $t_1 - t_2 = 0$. As to the latter, we state two theorems which are readily proved, but the proofs of which we will here omit. They are stated as follows:

Theorem I: The expression $x_1 y_3 - y_1 x_3$ can vanish only for the real values of $v_3 - v_1$.

Theorem II: In all cases where the times of the observations are distinct and where the difference of the longitudes of the comet in its orbit is not equal to an odd multiple of π, the expression (r_i, r_j) cannot vanish and the expressions $\dfrac{B_3}{B_1}, \dfrac{B_1}{A_1 B_3 - B_1 A_3}$ are legitimate fractions which may be expressed as series, each of which is convergent for all cases where $|t_3 - t_2| < \dfrac{p^{\frac{3}{2}}}{3}\sqrt{\tau_2^2 + 1}$ and $|t_1 - t_2| < \dfrac{p^{\frac{3}{2}}}{3}\sqrt{\tau_2^2 + 1}$. The first terms of these series are written out in the right members of (24). In many cases, however, the computer may prefer to use the fractions, and these are always the safer formulæ when doubt in any way exists as to their legitimacy.

17. *Computation by use of the series.*—The fractions on the right of (24) have both numerators and denominators in the form of series. Their radius of convergency is $R_t = \frac{p^{\frac{3}{2}}}{3}\sqrt{1+\tau_2^2}$, where $\tau_2 = \frac{3}{p^{\frac{3}{2}}}(t_2 - \Pi)$. If we make $k = \frac{1}{60}$, the following table will give corresponding maximum intervals of time for different values of p for which the series are convergent when t_2 is taken equal to Π:

TABLE I.

p	4	3	2.5	2.	1.5	1.25	1.	0.8	0.6	0.4	0.25	0.2	0.1	0.08	0.05	0.02
q	2	1.5	1.25	1.	0.75	0.62	0.5	0.4	0.3	0.2	0.12	0.1	0.05	0.04	0.02	0.01
da.,..............	160.0	103.4	79.8	56.0	36.2	27.4	20.0	14.1	9.0	5.0	2.5	1.8	0.6	0.44	0.22	0.06

It is evident that for any particular value of p the time intervals should be well within the limit of values for which the series are convergent. This is especially true if we would have the most rapid convergence—a thing most desirable from the standpoint of the computer. In fact, as is well known, it is imperative to have this convergence so rapid that at most but one or two terms will give sufficiently approximate values of the ratios. The reason for this is at once evident when we consider that the series are transcendental in character. Thus, the quantities $\frac{dr_2}{dt}$ and r_2 which enter into the terms higher than the first are essentially unknown from the start, and cannot even be guessed at with any degree of certainty until an approximate value of p has been obtained. It cannot be too strongly insisted upon, therefore, that, in order to get the closest determination of the ratios of the triangles, the greatest care must be taken to secure a set of time intervals which, by their co-ordination with the parameter of the orbit in hand, will make the series rapidly convergent. It is true that this is more or less a question of trial to start with; yet, when a value of p has been once computed by means of any set of time intervals, it will be seen at once whether the value so obtained is one for which the series are sufficiently convergent for the time intervals employed. If this is not the case, then new time intervals should be taken and the computation made over again.

PART II.

In this part of this paper are deduced differential equations of relation which give the errors of the computed elements of the cometary orbit expressed in terms of the errors of the observations. At the close is also given the results of a computation in which the formulæ are applied to an actual example.

18. *Geometrical relations.* — Let ρ represent the geocentric distance of the comet, λ and β its geocentric longitude and latitude, respectively. Let R represent the heliocentric distance of the earth, L and B the longitude and latitude of the sun, respectively. The subscripts 1, 2, 3 are used as before to represent the order of the observation to which the corresponding co-ordinate applies. With this notation any one of the following three equations will express the relation between the geocentric distances of the comet at the first and third observations. For their derivation the reader is referred to Moulton's *Introduction to Celestial Mechanics*, chap. x.

$$\rho_3 = m' + M' \rho_1 \, , \tag{1}$$

where m' and M' are defined by

$$m' = \frac{1}{\cos\beta_3 \sin(\lambda_3 - L_2) - \sin\beta_3 \cos\beta_2 \sin(\lambda_2 - L_2} \left[\frac{[r_2, r_3]}{[r_1, r_2]} L_1' + L_3' \right],$$

$$M' = \frac{[r_2, r_3] \sin\beta_1 \cos\beta_2 \sin(\lambda_2 - L_2) - \cos\beta_1 \sin(\lambda_1 - L_1)}{[r_1, r_2] \cos\beta_3 \sin(\lambda_3 - L_2) - \sin\beta_3 \cos\beta_2 \sin(\lambda_2 - L_2)} \, , \tag{1a}$$

$$L_1' = R_1 \sin(L_1 - L_2), \quad L_3 = R_3 = \sin(L_3 - L_2) \, .$$

$$\rho_3 = m'' + M'' \rho_1 \, , \tag{2}$$

where m'' and M'' are defined by

$$m'' = \frac{\sin\beta_2 \left[[r_2, r_3] R \cos(L_1 - L_2) + [r_1, r_2] R_3 \cos(L_3 - L_2) - [r_1, r_3] R_2 \right]}{[r_1, r_2] \left[\sin\beta_2 \cos\beta_3 \cos(\lambda_3 - L_2) - \sin\beta_3 \cos\beta \cos(\lambda_2 - L_2) \right]} \, ,$$

$$M'' = \frac{[r_2, r_3]}{[r_1, r_2]} \sin\beta_1 \cos\beta_2 \cos(\lambda_3 - L_2) - \sin\beta_3 \cos\beta_2 \cos(\lambda_2 - L_2^1) \, . \tag{2a}$$

$$\rho_3 = m''' + M''' \rho_1 \, , \tag{3}$$

where m''' and M''' are defined by

$$M''' = \frac{[r_2, r_3] \sin(\lambda_2 - \lambda_1) \cos\beta_1}{[r_1, r_2] \sin(\lambda_3 - \lambda_2) \cos\beta_3} \, , \tag{3a}$$

$$m''' = \frac{(r_2, r_3) R_1 \sin(L_1 - \lambda_1) + (r_1, r_2) R_3 \sin(L_3 - \lambda_3) - (r_1, r_3) R_2 \sin(L_2 - \lambda_2)}{[r_1, r_2] \cos\beta_3 \sin(\lambda_3 - \lambda_2)} \, .$$

19. *Generalities.* — Equations (1), (2), and (3) involve the dynamical law that the motion of a comet is in a plane passing through the center of the sun. The quantities m', m'', m''', M', M'', M''' are functions of the geocentric longitudes and latitudes λ_1, β_1, λ_2, etc., and of the ratios of the triangles. They also involve R_1, R_2, R_3, and the longitude and latitudes of the sun. But R_1, B_1, L_1, R_2, etc., are independent of the errors of observation, since they are taken from the Ephemeris. Likewise the time intervals which are involved in the ratios are independent of λ_1, β_1, λ_2, β_2, λ_3, β_3, which are derived directly from the right ascension and declination determined by the settings of the instrument. Owing to this last fact, we shall speak of λ_1, λ_2, β_1, etc., as observed co-ordinates.

In practice some one of the three equations (1), (2), (3) is usually more advantageous than the others owing to the particular problem of computation at hand. For the method of determining the one to be used, the reader is again referred to the same treatise and chapter as in the last article. For our purpose here, the fact just stated is of interest because it necessitates the derivation of a separate set of formulæ for the errors in the elements to correspond to each of the relations (1), (2), (3).

In addition to the relations already given, we shall have need of the following equation, due to Euler:

$$
\begin{aligned}
&(r_1 + r_3 + s)^{\frac{3}{2}} - (r_1 + r_3 - s)^{\frac{3}{2}} = 3(t_3 - t_1) \\
&s^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 \\
&\quad = \rho_1^2 - 2\rho_1 R_1 \cos\beta_1 \cos(\lambda_1 - L_1) + R_1^2 + 2\rho_1 R_3 \cos\beta_1 \cos(\lambda_1 - L_3) \qquad (4) \\
&\quad + \rho_3 - 2\rho_3 R_3 \cos\beta_3 \cos(\lambda_3 - L_3) + 2\rho_3 R_1 \cos\beta_3 \cos(\lambda_3 - L_1) \\
&\quad + 2R_1 R_3 \cos(L_3 - L_1) \; -
\end{aligned}
$$

where s is the chord connecting the first and third positions in the orbit. The quantities t_3, t_1 are given here in the units used in the first part of the paper; and it is understood that t_3 is larger than t_1, so that $t_3 - t_1$ is positive.

20. *Plan of procedure.* — From the relations already set forth, it is proposed to deduce formulæ expressing the variations, in p, Ω, i, ω and Π due to a variation in the observed co-ordinates λ_1, λ_2, λ_3, β_1, β_2, and β_3. In this work t_1, t_2, t_3 are considered constant, as are also the quantities R_1, R_2, R_3, L_1, L_2, L_3, which are obtained from the Ephemeris and depend directly on t_1, t_2, t_3. It is evident that r_1, r_2, r_3, ρ_1, ρ_2, ρ_3, v_1, v_2, v_3, which occur in the relations, are functions of λ_1, λ_2, λ_3, β_1, β_2, β_3, and will vary with these co-ordinates. Hence, incidentally,

formulæ are derived showing variations of these in terms of the observed co-ordinates. One thing further. The quantities $\frac{[r_2, r_3]}{[r_1, r_2]}$ and $\frac{[r_1, r_3]}{[r_1, r_2]}$ are functions of λ_1, λ_2, λ_3, β_1, etc.; but when these ratios are computed by means of series, it is easily seen that the above quantities enter only in the higher terms of the series—terms which are small and hence neglected on first approximation. It will remain to show that we may always neglect such terms in taking the partial differential coefficients of m', M', etc., in respect to the observed co-ordinates. This is taken up later.

It resolves itself to this, then: Each of the elements p, Ω, ω, i, and Π is a function of the six variables λ_1, λ_2, λ_3, β_1, β_2, β_3; but it is found that the work of derivation of the formulæ naturally divides itself into three parts. Thus we may get expressions to determine δp, $\delta\Omega$, $\delta\omega$, δi, and $\delta\Pi$ where these variations arise from a variation $\delta\lambda_1$ in λ_1 and $\delta\beta_1$ in β_1 of the first position. A second set of formulæ will determine δp, $\delta\Omega$, etc., where variations of like character are given to the observed co-ordinates of the second position; finally, a similar set are obtained for the third position. We take up the work in the order just indicated.

21. *The variations* $\delta\lambda_1$, $\delta\beta_1$.—The equations (1), (2), and (3) are all of the type

$$\rho_3 = m + M\rho_1 .$$

Hence we get for each of these

$$\delta\rho_3 = \frac{\partial m}{\partial\beta_1}\delta\beta_1 + \frac{\partial m}{\partial\lambda}\delta\lambda + \rho_1\frac{\partial M}{\partial\lambda_1}\delta\lambda_1 + M\delta\rho_1 , \tag{5}$$

where $\delta\rho_3$, $\delta\rho_1$ are changes in ρ_3, ρ_1 for the variations $\delta\beta_1$ and $\delta\lambda_1$ in the arguments β_1 and λ_1. From $(1)_a$, $(2)_a$, $(3)_a$ we get the following expressions for the partial derivatives in (5), where we count the ratios of the triangles as independent of β_1 and λ_1 according to the remarks of the previous article.

$$\frac{\partial m}{\partial\beta_1} = 0 , \quad \frac{\partial m}{\partial\lambda_1} = 0 . \tag{6}$$

$$\begin{aligned} \frac{\partial M'}{\partial\beta_1} &= \frac{[r_2, r_3]}{[r_1, r_2]}\frac{\cos\beta_1\cos\beta_2\sin(\lambda_2 - L_2) + \sin\beta_1\sin\beta_2\sin(\lambda_1 - L_2)}{\sin\beta_2\cos\beta_3\sin(\lambda_3 - L_2) - \sin\beta_3\cos\beta_2\sin(\lambda_2 - L_2)} , \\ \frac{\partial M'}{\partial\lambda_1} &= \frac{[r_2, r_3]}{[r_1, r_2]}\frac{\sin\beta_2\cos\beta_1\cos(\lambda_1 - L_2}{\sin\beta_2\cos\beta_3\sin(\lambda - L_2) - \sin\beta_3\cos\beta_2\sin(\lambda_2 - L_2)} , \end{aligned} \tag{7}$$

$$\frac{\partial M''}{\partial\beta_1}=\frac{[r_2,\ r_3]}{[r_1,\ r_2]}\frac{\cos\beta_1\cos\beta_2\cos(\lambda_2-L_2)+\sin\beta_2\cos(\lambda_2-L_2}{\sin\beta_2\cos\beta_3\cos(\lambda_3-L_3)-\sin\beta_3\cos\beta_2\cos(\lambda_2-L_2)}\,,$$
$$\frac{\partial M''}{\partial\lambda_1}=\frac{[r_2,\ r_3]}{[r_1,\ r_2]}\frac{\cos\beta_1\sin\beta_2\sin(\lambda_1-L_0)}{\sin\beta_2\cos\beta_3\cos(\lambda_3-L_3)-\sin\beta_3\cos\beta_2\cos(\lambda_2-L_2)}\,, \tag{8}$$

$$\frac{\partial M'''}{\partial\beta_1}=\frac{[r_2,\ r_3]}{[r_1,\ r_2]}\frac{\sin\beta_1\ \sin(\lambda_1-\lambda_2)}{\cos\beta_3\sin(\lambda_3-\lambda_2)}\,.$$
$$\frac{\partial M'''}{\partial\lambda_1}=\frac{[r_2,\ r_3]}{[r_1,\ r_2]}\frac{\cos\beta_1\cos(\lambda_1-\lambda_2)}{\cos\beta_3\sin(\lambda_1-\lambda_2)}\,. \tag{9}$$

Using the second relation of (4), we obtain

$$\delta s=\frac{\partial s}{\partial\beta_1}\delta\beta_1+\frac{\partial s}{\partial\lambda_1}\delta\lambda_1+\frac{\partial p}{\partial\rho_1}\delta\rho_1+\frac{\partial s}{\partial\rho_3}\delta\rho_3\,, \tag{10}$$

where we have

$$\frac{\partial s}{\partial\rho_1}=\frac{1}{s}[\rho_1-R_1\cos\beta_1\cos(\lambda_1-L_1)+R_3\cos\beta_1\cos(\lambda_1-L_3)]\ ;$$
$$\frac{\partial s}{\partial\rho_3}=\frac{1}{s}[\rho_3-R_3\cos\beta_3\cos(\lambda_3-L_3)+R_1\cos\beta_3\cos(\lambda_3-L_1)]\ ;$$
$$\frac{\partial s}{\partial\beta_1}=\frac{1}{s}[\rho_1R_1\sin\beta_1\cos(\lambda_1-L_1)-\rho_1R_3\sin\beta_1\cos(\lambda_1-L_3)]\ ; \tag{11}$$
$$\frac{\partial s}{\partial\lambda_1}=\frac{1}{s}[\rho_1R_1\cos\beta_1\sin(\lambda_1-L_1)-\rho_1R_3\cos\beta_1\sin(\lambda_1-L_3)]\ ;$$

By means of (5) we may eliminate $\delta\rho_3$ from (10), and get

$$\delta s=\frac{\partial s}{\partial\beta_1}+\rho_1\frac{\partial M}{\partial\beta_1}\frac{\partial s}{\partial\rho_3}\delta\beta_1+\left(\frac{\partial s}{\partial\lambda_1}+\rho_1\frac{\partial M}{\partial\lambda_1}\frac{\partial s}{\partial\rho_3}\right)\delta\lambda+\frac{\partial s}{\partial\rho_1}$$
$$+M\left(\frac{\partial s}{\partial\rho_3}\right)\delta\rho_1\,. \tag{12}$$

From the first relation of (4) we get, where for brevity we put $r_1+r_3=K$;

$$\delta K=\frac{\sqrt{K+s}+\sqrt{K-s}}{\sqrt{K+s}\quad\sqrt{K-s}}\delta p\,. \tag{13}$$

This equation may be written

$$\delta K=-\frac{K+K\sqrt{1-\left(\frac{s}{K}\right)}}{s}\delta s\,.$$

Now, where $v_3 - v_1 < 180°$, which includes all practical cases in which it is possible to make use of the expansions treated of in Part I, $\frac{s}{K}$ will always be less than unity; and hence we may write

$$\sqrt{1-\left(\frac{s}{K}\right)^2} = 1 - \frac{1}{2}\left(\frac{s}{K}\right)^2 - \frac{1}{2} : \frac{1}{4}\left(\frac{s}{K}\right)^4 \cdots \frac{1}{n} : \frac{3.5 \cdots (2n-3)}{2n}\left(\frac{s}{K}\right)^{2n} .$$

From this it follows that (13) may be written

$$\delta K = -\frac{2K}{s}\,\delta s + \left[\frac{1}{2}\frac{s}{K} + \frac{1}{2} : \frac{1}{4}\left(\frac{s}{K}\right)^3 + \cdots \frac{1}{n} : \frac{3.5 \cdots (2n-3)}{2n}\left(\frac{s}{K}\right)^{2n-1}\right]\delta s \,. \tag{14}$$

The series on the right will, in all legitimate cases, be very convergent, owing to the fact that there must always be the strong inequality $s < K$. In general not more than three terms will be necessary (usually two are sufficient) where six-place logarithms are used.

By use of (12 and 14) we are able to express $\delta(r_1 + r_3)$ directly in terms of variations of β_1, λ_1, and ρ_1. We do not write out the result, which is very simple. It may be noticed here that in (14) a singularity enters into the coefficient of δs when s approaches zero. In this case the value of δK would depend on the first term on the right and would be large. The same difficulty is found in the value of δs itself, since, as seen from (11), the partial derivatives in respect to ρ_1, ρ_3, λ_1, β_1 each become large. From this we conclude that *when the observations are taken at very short intervals in the orbit, then the computed value of $r_1 + r_3$ will be very inaccurate, owing to the errors of the observations.*

From a well-known trigonometrical relation upon the triangle whose sides are ρ, r, and R we have the two relations for the first and third positions of the comet and earth in respect to the sun:

$$\begin{aligned} r_1^2 &= \rho_1^2 - 2\rho_1 R_1 \cos\beta_1 \cos(\lambda_1 - L_1) + R_1^2 \,, \\ r_3^2 &= \rho_3^2 - 2\rho_3 R_3 \cos\beta_3 \cos(\lambda_3 - L_3) + R_3^2 \,. \end{aligned} \tag{15}$$

From (15) we obtain

$$\begin{aligned} \delta r_1 &= \frac{\partial r_1}{\partial \rho}\,\delta\rho_1 + \frac{\partial r_1}{\partial \beta_1}\,\delta\beta_1 + \frac{\partial r_1}{\partial \lambda_1}\,\delta\lambda_1 \,, \\ \delta r_3 &= \frac{\partial r_3}{\partial \rho_3}\,\delta\rho_3 \,, \end{aligned} \tag{16}$$

where we have

$$\left.\begin{aligned}
\frac{\partial r_1}{\partial \beta_1} &= \frac{1}{r}\left[\rho_1 R_1 \sin \beta_1 \cos (\lambda - L_1)\right] , \\
\frac{\partial r_1}{\partial \lambda^1} &= \frac{1}{r}\left[\rho_1 R_1 \cos \beta_1 \sin (\lambda_1 - L_1)\right] . \\
\frac{\partial r_1}{\partial \rho_1} &= \frac{1}{r_1}\left[\rho_1 - R_1 \cos \beta_1 \cos (\lambda_1 - L_1)\right] , \\
\frac{\partial r_3}{\partial \rho_3} &= \frac{1}{r_3}\left[\rho_3 - R_3 \cos \beta_3 \cos (\lambda_3 - L_3)\right] .
\end{aligned}\right\} \tag{17}$$

By means of (5) equations (16) may be written

$$\begin{aligned}
\delta r_1 &= \frac{\partial r_1}{\partial \rho_1}\delta\rho_1 + \frac{\partial r_1}{\partial \beta_1}\delta\beta_1 + \frac{\partial r}{\partial \lambda}\delta\lambda_1 . \\
\delta r_3 &= \left[\rho_1 \frac{\partial M}{\partial \beta_1}\delta\beta_1 + \rho_1 \frac{\partial M}{\partial \lambda_1}\delta\lambda_1 + M\delta\rho_1\right]\frac{\partial r_3}{\partial \rho_3} .
\end{aligned} \tag{18}$$

Now from (14), when combined, as suggested above, with (12), we have

$$\begin{aligned}
\delta r_3 &= \delta K - \delta r_1 , \\
\delta r_1 &= \delta K - \delta r_3 ,
\end{aligned} \tag{19}$$

where δK involves only the differentials $\delta\beta_1$, $\delta\lambda_1$, $\delta\rho_1$ in linear expressions. Hence, by combining (18) and (19) we may get δr_3 and δr_1 in expressions of the form :

$$\begin{aligned}
\delta r_1 &= c\,\delta\beta_1 + c'\,\delta\lambda_1 , \\
\delta r_3 &= d\,\delta\beta_1 + d'\,\delta\lambda_1 ,
\end{aligned} \tag{20}$$

where c, c', d, d' are constants. By substituting these back into (18) we may also get $\delta\rho_1$ in the form

$$\delta\rho_3 = e\,\delta\beta_1 + e'\,\delta\lambda_1 , \tag{21}$$

where e and e' are constants not involving $\delta\beta_1$ or $\delta\lambda_1$. If we place this last in (5), we also get $\delta\rho_3$ in the form

$$\delta\rho_3 = f\,\delta\beta_1 + f'\,\delta\lambda_1 , \tag{22}$$

where f, f' do not involve $\delta\beta_1$ or $\delta\lambda_1$. These auxiliary relations, δr_1, δr_3, $\delta\rho_1$, $\delta\rho_3$, are very useful for the work which follows in getting the variations of the elements p, Ω, ω, i, and Π in terms of $\delta\beta_1$, $\delta\lambda_1$.

We start with the equations (7) of Part I, as follows :

$$
\left.\begin{aligned}
p &= r\,(1 + \cos v) = 2r\cos^2\frac{v}{2}\,,\\
\tan\frac{v}{2} + \frac{1}{3}\tan^3\frac{v}{2} &= 2\,\frac{(t-\Pi)}{p^{\frac{3}{2}}}\,.
\end{aligned}\right\}\qquad(23)
$$

From the first of these we get for the first and third positions

$$
\tan\frac{v_1}{2} = \pm\sqrt{\frac{2r_1 - p}{p}}\,,\quad \tan\frac{v_3}{2} = \pm\sqrt{\frac{2r_3 - p}{p}}\,,
$$

where the radicals are to be taken positive, if $\frac{v}{2}$ is the first or third quadrant; and negative if in the second or fourth quadrant. By use of these relations we obtain from the second relation of (23)

$$
\left.\begin{aligned}
\sqrt{\frac{2r_1 - p}{p}} + \frac{1}{3}\sqrt{\left(\frac{2r_1 - p}{p}\right)^3} &= 2\,(t_1 - \Pi)\,,\\
\sqrt{\frac{2r_3 - p}{p}} + \frac{1}{3}\sqrt{\left(\frac{2r_3 - p}{p}\right)^3} &= 2\,(t_3 - \Pi)\,.
\end{aligned}\right\}\qquad(24)
$$

Eliminating Π from (24) and clearing of fractions, we get

$$
3p\left[\sqrt{2r_3 - p} - \sqrt{2r - p} + \sqrt{(2r_3 - p)^3} - \sqrt{(2r_1 - p)^3}\right.
$$
$$
\left. = 6(t_3 - t_1)\right]\,. \qquad (25)
$$

By giving variations δr_1 and δr_3 to r_1 and r_3 respectively, we obtain, after collecting and simplifying and solving for δp,

$$
\delta p = \frac{2r_1\sqrt{2r_3 - p}}{(r_3 - p)\sqrt{2r - p} - (r_1 - p)\,2r_3 - p}\,\delta r_1
$$
$$
- \frac{2r_3\sqrt{2r - p}}{(r_1 - p)\sqrt{2r_1 - p} - (r_1 - p)\sqrt{2r_3 - p}}\,\delta r_3\,. \qquad (26)
$$

In (26) it must be kept in mind that the radicals are positive or negative according as $\tan\frac{v}{2}$ is positive or negative. This equation, when the auxiliary equations (20) are used, gives δp as a linear function of $\delta\beta_1$, $\delta\lambda_1$.

From the relations

$$
p = r_1\,(1 + \cos v_1)\,,
$$
$$
p = r_3\,(1 + \cos v_3)\,,
$$

we derive by differentiation

$$\delta p = \frac{p}{r_1}\,\delta r_1 - r_1 \sin v_1 \delta v_1 \ ,$$

$$\delta p = \frac{p}{r_3}\,\delta r_3 - r_3 \sin v_3 \delta v_3 \ ,$$

from which

$$\left.\begin{aligned} \delta v_1 &= \frac{p}{r_1^2 \sin v_1}\,\delta r_1 - \frac{\delta p}{r_1 \sin v_1} \ , \\ \delta v_3 &= \frac{p}{r_3^2 \sin v_3}\,\delta r_3 - \frac{\delta p}{r_3 \sin v_3} \ . \end{aligned}\right\} \tag{27}$$

By means of (27), (26), and (20), we may get δv_1, δv_3 expressed as linear functions of $\delta\beta_1$, $\delta\lambda_1$.

If we denote the argument of latitude by $u \equiv v + \omega$, we have then the relation

$$u_3 - u_1 = v_3 - v_1 \ ;$$

from which

$$\delta u_2 - \delta u_1 = \delta v_3 - \delta v_1 \ . \tag{28}$$

From (6) of Part I and corresponding relations expressing the cartesian co-ordinates in terms of geocentric longitude and latitudes, we get

$$\left.\begin{aligned} \delta z_1 &= \frac{z_1}{r_1}\,\delta r_1 + r_1 \sin u_1 \cos i \delta i + r_1 \cos u_1 \sin i \delta u_1 \ , \\ \delta z_1 &= \sin \beta \delta \rho_1 + \rho_1 \cos \beta_1 \delta \beta_1 \ , \end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} \delta z_3 &= \frac{z_3}{r_3}\,\delta r_3 + r_3 \sin u_3 \cos i \delta i + r_3 \cos u_3 \sin i \delta u_3 \ , \\ \delta z_3 &= \sin \beta_3 \delta \rho_3 \ . \end{aligned}\right\} \tag{30}$$

By means of (28), (29), and (30), we may get δi, δu_1, δu_3, in linear functions of $\delta\beta_1$, $\delta\lambda_1$.

We also obtain

$$\left.\begin{aligned} \delta x_1 &= \frac{x_1}{r_1}\,\delta r_1 - y_1 \delta\Omega + z_1 \sin \Omega \delta i - r_1 (\sin u_1 \sin \Omega - \cos u_1 \cos \Omega \cos i)\,\delta u_1 \ , \\ \delta x_1 &= \cos \beta_1 \cos \lambda_1 \delta \rho_1 - \rho_1 \sin \beta_1 \cos \lambda_1 \delta \beta_1 - \rho_1 \cos \beta_1 \sin \lambda_1 \delta \lambda_1 \ . \end{aligned}\right\} \tag{31}$$

By means of these equations, or the corresponding ones for the third position which may be used as a check, we may get $\delta\Omega$.

From the relation $\omega = u - v$, we obtain

$$\left.\begin{aligned} \delta\omega &= \delta u_1 - \delta v_1 \ , \\ \delta\omega &= \delta u_3 - \delta v_3 \ . \end{aligned}\right\} \tag{32}$$

Finally from

$$\tan\frac{v}{2}+\frac{1}{3}\tan\frac{3v}{2}=\frac{2(t-\Pi)}{p_2^3}\ ,$$

we obtain

$$\begin{aligned}\delta\Pi&=-\frac{3}{2p}(t_1-\Pi)\,\delta p-\frac{p_2^3}{4}\sec^4\frac{v_1}{2}\,\delta v_1\ ,\\ \delta\Pi&=-\frac{3}{2p}(t_3-\Pi)\,\delta p-\frac{p_2^3}{4}\sec^4\frac{v_3}{2}\,\delta v_3\ .\end{aligned}\tag{33}$$

With these we have a complete set of formulæ by which the variation of any one of the five elements of the cometary orbit is expressible as a linear function of $\delta\beta_1$ and $\delta\lambda_1$.

22. *The variations* $\delta\beta_2\ \delta\lambda_2$.—If now errors are made in the observation of the second place, these errors will also have an effect upon the elements. We consider the errors thus caused and give formulæ for their computation. In these formulæ it is to be noticed that we use the expressions $\delta\rho_1$, $\delta\rho_3$, δr_1, etc., to designate the variation of the quantities ρ_1, ρ_3, r_1, etc., only so far as β_2 and λ_2 are concerned. They must not be confused with the same expressions used heretofore, where only λ_1 and β_1 were considered to vary. The apparent ambiguity is justified by the simplicity which this usage gives to the writing of the two sets of formulæ. Furthermore, we content ourselves here by simply writing down the results of the derivations. This is done because the actual work of obtaining the equations is very similar to that pursued in the last article, and where any divergence occurs the formulæ themselves enable one to see the method used. Attention is again called to the approximation used in the values of the ratios of the triangles in obtaining the partial derivatives of M and m in respect to β_2 and λ_2.

To start with we have the equation analogous to (5):

$$\delta\rho_3=M\delta\rho_1+\frac{\partial m}{\partial\beta_2}\delta\beta_2+\frac{\partial m}{\partial\lambda_2}\delta\lambda_2+\rho_1\frac{\partial M}{\partial\beta_2}\delta\beta_2+\rho_1\frac{\partial M}{\partial\lambda_2}\delta\lambda_2\ ,\tag{34}$$

where the partial derivatives are gotten from the expressions for m', m'', m''', M', M'', M''' given in (1) *a*, (2) *a*, and (3) *a;* owing to their length we omit them here. It is to be noticed, however, that here $\dfrac{\partial M'''}{\partial\beta_2}$ is zero.

$$\begin{aligned}\delta s=&\frac{1}{s}[\rho-R_1\cos\beta_1\cos(\lambda_1-L_1)+R_3\cos\beta_1\cos(\lambda_1-L_3)]\,\delta\rho_1\\ &+\frac{1}{s}[\rho_3-R_3\cos\beta_3\cos(\lambda_3-L_3)+R_1\cos\beta_3\cos(\lambda_3-L_1)]\,\delta\rho_3\ .\end{aligned}\tag{35}$$

$$\delta K = -\frac{2K}{s}\delta s + \left[\frac{1}{2}\frac{s}{K} + \cdots \frac{1}{\underline{|n}}\frac{3\cdot 5\cdots(2^n-3)}{2^n}\left(\frac{s}{K}\right)^{2n-1} + \cdots\right]\delta s\ , \tag{36}$$

where $K = r_1 + r_3$.

$$\begin{aligned}\delta r_1 &= \frac{\delta\rho_1}{r_1}[\rho_1 - R_1\cos\beta_1\cos(\lambda_1 - L_1)]\ ,\\ \delta r_3 &= \frac{\delta\rho_3}{r_3}[\rho_3 - R_3\cos\beta_3\cos(\lambda_3 - L_3)]\ .\end{aligned} \tag{37}$$

$$\begin{aligned}\delta r_1 &= \delta K - \delta r_3\ ,\\ \delta r_3 &= \delta K - \delta r_1\ .\end{aligned} \tag{38}$$

$$\delta p = \frac{2r_1\sqrt{2r_1 - p}}{(r_3 - p)\sqrt{2r_1 - p} - (r_1 - p)\sqrt{2r_3 - p}}\,\delta r_1 - \frac{2r_3\sqrt{2r_1 - p}}{(r_3 - p)\sqrt{2r_1 - p} - (r_1 - p)\sqrt{2r_3 - p}}\,\delta r_3\ . \tag{39}$$

$$\begin{aligned}\delta v_1 &= \frac{p}{r_1^2\sin v_1}\,\delta r_1 - \frac{\delta p}{r_1\sin v_1}\ ,\\ \delta v_3 &= \frac{p}{r_3^2\sin v_3}\,\delta r_3 - \frac{\delta p}{r_3\sin v_3}\ ,\end{aligned} \tag{40}$$

$$\delta u_3 - \delta u_1 = \delta v_3 - \delta v_1\ . \tag{41}$$

$$\begin{aligned}\delta z_1 &= \frac{z_1}{r_1}\delta r_1 + r_1\sin u_1\cos i\,\delta i + r_1\cos u_1\sin i\,\delta u_1\ ,\\ \delta z_1 &= \sin\beta_1\delta\rho_1\ .\end{aligned} \tag{42}$$

$$\begin{aligned}\delta z_3 &= \frac{z_3}{r_3}\delta r_3 + r_3\sin u_3\cos i\,\delta i + r_3\cos u_3\sin i\,\delta u_3\ ,\\ \delta z_3 &= \sin\beta_3\delta\rho_3\ .\end{aligned} \tag{43}$$

$$\begin{aligned}\delta x_1 &= \frac{x_1}{r_1}\delta r_1 - y_1\delta\Omega + z_1\sin\Omega\,\delta i - r_1\,[\sin u_1\sin\Omega - \cos u_1\cos\Omega\cos i]\,\delta u_1\ ,\\ \delta x_1 &= \cos\beta_1\cos\lambda_1\,\delta\rho_1\ .\end{aligned} \tag{44}$$

$$\begin{aligned}\delta\omega &= \delta u_1 - \delta v_1\ ,\\ \delta\omega &= \delta u_3 - \delta v_3\ .\end{aligned} \tag{45}$$

$$\begin{aligned}\delta\Pi &= -\frac{3}{2p}(t_1 - \Pi)\,\delta p - \frac{p^{\frac{3}{2}}}{4}\sec^4\frac{v_1}{2}\,\delta v_1\ ,\\ \delta\Pi &= -\frac{3}{2p}(t_3 - \Pi)\,\delta p - \frac{p^{\frac{3}{2}}}{4}\sec^4\frac{v_3}{2}\,\delta v_3\ .\end{aligned} \tag{46}$$

$$\begin{aligned}\delta\varpi &= \delta u_1 + \delta\Omega - \delta v_1\ ,\\ \delta\varpi &= \delta u_3 + \delta\Omega - \delta v_3\ .\end{aligned} \tag{47}$$

This completes the list of formulæ for obtaining the variations of the elements in terms of the variations $\delta\beta_2$, $\delta\lambda_2$.

23. *The variations* $\delta\beta_3$, $\delta\lambda_3$.—It is easily seen that the formulæ in the case of errors in the third position will be very similar to those of the first position. This is due to the fact that in the computation of the elements the co-ordinates of the first and last positions play very similar rôles. We do not give the formulæ for the last position here, since it would unnecessarily prolong this discussion.

24. *Dependence of the ratios of the triangles on* λ_1, β_1, λ_3, β_3.—We come now to a question left over from a previous article. It is as to the dependence of the ratios of the triangles upon the co-ordinates of the first and third positions. The question reduces to one as to the dependence of r_2 and $\frac{dr_2}{dt}$ upon the above co-ordinates. That this is true is foreshadowed by the terms of the series written out on the right of (24), Part I; but it is also capable of derivation from the Newtonian law of motion itself that such is the case. For we have in general

$$\frac{d^2r}{dt^2} = \frac{1}{r^2};$$

and by means of this relation all the higher derivations of r_2 in the development of the ratios are reducible back upon r_2 and $\frac{dr_2}{dt}$.

Now, the quantities r_2 and $\frac{dr_2}{dt}$ are by the manner of their computation in Olbers's method each functions of β_1, λ_1, β_3, λ_3 as well as of λ_2, or β_2, or both of the latter two, according to the particular one of the equations (1), (2), or (3) which have been employed. But, in taking the partial derivations of M and m in respect to the co-ordinates λ_1, β_1, λ_3, etc., we have assumed the ratios to be independent of these quantities. It is necessary, then, to justify this assumption.

To begin with it can be justified only from the standpoint of its being a near approximation to the truth when the computation has been carried out in the method described in article 17, namely, when the time intervals have been so chosen as to give series (24), Part I, the proper convergency. As we have already remarked, this convergency should be so rapid that for any given case the remainders after the second term will be of the order of smallness of the lowest decimal place which is omitted in the process of computation. In order to

verify the statement made above as to the validity of the approximation in question, we prove the following theorem, which we then proceed to apply.

Theorem: The variation δr_2 arising from uncertainties in λ_1, λ_2, λ_3, β_1, β_2, β_3 is at least of the order of smallness of δp, and $\delta\Pi$.

For, in the same manner as we obtained (33), we get, after reduction,

$$\delta v_2 = \frac{3(\Pi - t_2)}{2r_2^2 p^{\frac{1}{2}}}\,\delta p - \frac{p^{\frac{1}{2}}}{r_2^2}\,\delta\Pi\,. \tag{48}$$

From the relation $r_2(1 + \cos v_2) = p$, we get

$$\frac{p}{r_2}\,\delta r_2 - r_2 \sin v_2\,\delta v_2 = \delta p\,;$$

whence, by use of (48),

$$\delta r_2 = \frac{r_2}{p}\,\delta p + \frac{3}{2p^{\frac{3}{2}}}\left[(\Pi - t_2)\sin v_2 - \frac{\sin v_2}{p^{\frac{1}{2}}}\right]\delta\Pi\,. \tag{49}$$

An inspection of the right member of (49) at once makes the truth of the theorem apparent.

Now, in getting the partial derivatives in (7), (8), (9), terms are omitted, among others, of the following forms:

$$c_1\,\frac{(t_3 - t_2)^2 - (t_3 - t_1)^2}{r_2^4}\,\delta r_2\,,\quad c_2\,\frac{(t_3 - t_2)^3 + (t_2 - t_1)^3}{r_2^5}\frac{dr_2}{dt}\,\delta r_2\,,\ \text{etc.}\ —$$

where c_1, c_2 are quantities easily determined from equations (1), (2), (3), and which are small when the problem is taken under proper conditions. Now, for any particular and approximate set of quantities $(t_3 - t_2)$, $(t_2 - t_1)$, and p, it is known that the coefficients of (50), aside from the factors c_1, c_2, are small — not amounting to more than a digit in the third or fourth decimal place. For the discussion of this in a particular case, see *Moulton's Introduction to Celestial Mechanics*, art. 208. If we accept these statements, which are shown to be true by mere mechanical computation, it results that the terms omitted in art. 21 are not in general of appreciable size. In any case where the formulæ of this paper are used for computation, the value of these terms may be computed after the manner of Dr. Moulton's discussion just cited, and their probable value obtained. Should they be of considerable size, they must be taken into account in the computation of the uncertainties of the elements.

25. *Computation of the errors.—An example:* The following application of the formulæ (1) to (33) is here given in abstract as an example to accompany the theory here set forth. The computation of the elements was made by Dr. K. Laves and members of his class in the summer of 1900. The remaining computation was done by myself. Most of the results were checked by various devices invented for the occasion, some of which are mentioned in the previous discussions.

The orbit was that of the comet Borelly-Brooks (1900.6). The observations were made at the Chamberlain Observatory of the University of Denver by Mr. Ling with the twenty-inch refractor. They were as follows:

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$\lambda = 6^h\ 59^m\ 47^s.63$, $\phi = 39°\ 40'\ 36''.4$

	Comp.*	No. Comp.	α ☉ $- \alpha$ *	Greenwich M. T.
1900, July $24^d\ 13^h\ 37^m\ 21^s$	Lamb 1541	20 ; 6	$16'\ 47''.2$	$20^h\ 37^m\ 8^s.63$
August $2^d\ 12^h\ 9^m\ 56^s$	Bonn 2544	20 ; 8	$2'\ 28''.4$	$19^h\ 9^m\ 43^s.63$
August $6^d\ 10^h\ 26^m\ 16^s$	Lamb (u3) 1385	20 ; 6	$2'\ 27''.9$	$17^h\ 26^m\ 3^s.63$

Elements computed by class and Dr. Laves:

$$\log q = 0.006363$$
$$\Omega = 327°\ 59'\ 59''$$
$$i = 62°\ 29'\ 32''$$
$$\omega = 12°\ 24'\ 47''$$
$$\frac{1}{k}\,\Pi = \text{August 3, 1900, Greenwich M. T.}$$

The following values were obtained for co-ordinates by Dr. Laves, which I have corrected for the notation used in this paper and used throughout in computation:

$$\lambda_1 = 49°\ 13'\ 22''.6\,, \quad \beta_1 = 13°\ 28'\ 19''\,,$$
$$\lambda_2 = 54°\ 20'\ 46''.0\,, \quad \beta_2 = 24°\ 40'\ 31''\,,$$
$$\lambda_3 = 60°\ \ 4'\ 43''.0\,, \quad \beta_3 = 35°\ \ 1'\ 56''\,,$$
$$\log M' = 9.950021 - 10\,, \quad \log (r_1 + r_3) = 0.308352\,,$$
$$\log\ r_1 = 0.007514\,, \quad \log s = 9.278413 - 10\,.$$
$$\log\ r_3 = 0.007130\,,$$

We have numbered the equations giving the results for each step so

as to correspond with the formulæ from which they are derived in art. 21. Where the coefficients are logarithms, we have indicated it by placing the left-hand member in parenthesis, thus (δr_1), (δr_3), etc.

$$\delta\rho_3 = 1.05854\,\delta\rho_1 \quad - 2.26677\,\delta\beta_1 \quad - 0.147043\,\delta\lambda_1\,. \tag{5'}$$

$$\left.\begin{aligned} \delta s &= 3.49690\,\delta\rho_1 \quad - 3.77947\,\delta\beta_1 \quad + 0.20662\,\delta\lambda_1\,. \\ (\delta s) &= 0.543683\,\delta\rho_1 - 0.577421\,\delta\beta_1 + 9.315172\,\delta\lambda_1\,. \end{aligned}\right\} \tag{12'}$$

$$\delta K = -\,74.76678\,\delta\rho_1 + 80.80672\,\delta\beta_1 - 4.41772\,\delta\lambda_1\,. \tag{14'}$$

$$\left.\begin{aligned} \delta r_1 &= 0.236288\,\delta\rho_1 - 0.577421\,\delta\beta_1 + 0.032823\,\delta\lambda_1\,, \\ \delta r_3 &= 0.2106228\,\delta\rho_1 + 0.022165\,\delta\beta_1 - 0.416143\,\delta\lambda_1\,. \end{aligned}\right\} \tag{18'}$$

$$\left.\begin{aligned} \delta r_3 &= -\,74.97740\,\delta\rho_1 + 80.7845\,\delta\beta_1 \quad - 4.00158\,\delta\lambda_1\,, \\ \delta r_1 &= -\,75.00307\,\delta\rho_1 + 81.31272\,\delta\beta_1 - 4.38489\,\delta\lambda_1\,. \end{aligned}\right\} \tag{19'}$$

From (18′) and (19′) we get one and the same result, as follows:

$$\delta\rho_1 = 108080\,\delta\beta_1 - 0.052766\,\delta\lambda_1\,. \tag{5''}$$

Putting this in (5′)

$$\delta\rho_3 = -\,1.12271\,\delta\beta_1 - 0.202898\,\delta\lambda_1\,.$$

Putting these in (18′) and (19′),

$$\left.\begin{aligned} \delta r_1 &= 0.24981\,\delta\beta_1 - 0.42726\,\delta\lambda_1\,, \\ \delta r_3 &= -\,0.25062\,\delta\beta_1 - 0.04530\,\delta\lambda_1\,. \end{aligned}\right\} \tag{19''}$$

The uncertainty in p was found to be

$$\delta p = -\,0.058051\,\delta\beta_1 - 0.437117\,\delta\lambda_1\,. \tag{26'}$$

The following then were obtained in the order given:

$$\left.\begin{aligned} \delta v_1 &= -\,5.31986\,\delta\beta_1 + 3.96961\,\delta\lambda_1\,, \\ \delta v_3 &= -\,5.18735\,\delta\beta_1 + 4.06573\,\delta\lambda_1\,. \end{aligned}\right\} \tag{27'}$$

$$\delta u_3 - \delta u_1 = 0.13251\,\delta\beta_1 + 0.09612\,\delta\lambda_1\,. \tag{28'}$$

$$\delta i = -\,15.0415\,\delta\beta_1 - 2.4690\,\delta\lambda_1\,. \tag{30'}$$

$$\left.\begin{aligned} \delta u_1 &= 1.6237\,\delta\beta_1 + 0.18104\,\delta\lambda_1\,, \\ \delta u_3 &= 1.75628\,\delta\beta_1 + 0.2772\,\delta\lambda_1\,. \end{aligned}\right\} \tag{30''}$$

$$\delta\Omega = 0.6799\,\delta\beta_1 - 0.0763\,\delta\lambda_1\,. \tag{31'}$$

$$\frac{1}{k}\,\delta\Pi = -\,221.\ \delta\beta_1 + 170.\ \delta\lambda_1\,. \tag{33'}$$

In (33′) the digit in units place is uncertain by 3 in each coefficient. We finally get for the uncertainty in the longitude of the perihelion

$$\delta\omega = 6.9436\,\delta\beta_1 - 3.7886\,\delta\lambda_1\,. \tag{32'}$$

In order to get the amount of an error of an element for a given error in an observed co-ordinate, we need only substitute the values of $\delta\beta_1$ and $\delta\lambda_1$ which are allowable from the nature of the conditions under which the settings of the instrument were made. Suppose, for instance, that $\delta\beta_1$ equals to one second of arc. This, in circular measure, will correspond to 0.00000485. Then by (33′) an error of one unit in the third decimal place would result; while by (27′) the error would be confined to the seventh decimal place, and in this latter case be perfectly harmless for ordinary six-place tables.

As to the amount of error actually probable, various observers differ in their estimates. Le Verrier says:[1] "Experience has proved that, for stars with feeble light, errors of 4″ or 5″ do not exceed the limits of the possible nor indeed of the probable." In this he was referring to the observation of the asteroids. Dr. Hussey, of the Lick Observatory, considers this estimate far too large for the case where a star and an asteriod are being compared with large modern instruments. However he states that "for comets the *conditions vary greatly;* and for those without visible nuclei, large and cloudlike; in such cases, even under favorable atmospheric and instrumental conditions, " one might be doing very well indeed if he kept his errors of observation below 10." Dr. Barnard, of Yerkes Observatory, estimates, with Dr. Hussey, that 0″.1 ought to be the limit of error for an asteroid of the fainter kind if the conditions are favorable; but he says: "A comet's position is far more uncertain; and discordances of *several seconds of arc* are not unusual in the work of good observers. Much depends, of course, on the presence or absence of a nucleus and the faintness of the comet; but in general comet observations are *distressfully discordant.*"

The above statements of Dr. Hussey and Dr. Barnard are from personal letters to the writer in answer to inquiries in regard to the accuracy of comet observations with the best modern instruments. If we admit, then, that the errors in observation may vary from one to ten seconds of arc in the case of comets, the results (5′) to (32′) show that the computed elements will be discordant, owing to such errors in a single co-ordinate, as follows:

Element p is discordant in sixth or seventh decimal place,
Element i is discordant in fourth or fifth decimal place,
Element Ω is discordant in fifth or sixth decimal place,
Element Π is discordant in third or fourth decimal place,
Element ω is discordant in fourth or fifth decimal place.

[1] *C. R.*, Vol. XXV, p. 573.

These results might be very much decreased when combined with like errors in the measurements of the other five observed co-ordinates; or they might so balance each other that very little discordance would be present in the computed elements. As to this, however, we can only say that they are still uncertain by the amount represented by the combined effect of all the errors arising from all of the observed elements.

BIOGRAPHICAL.

William Albert Hamilton was born May 9, 1869, near Zanesville, Ind., and received his elementary education in the district schools, which he attended during the winter months. He was prepared for college at Roanoke Classical Seminary of Roanoke, Ind., and in the preparatory department of North Manchester College, of North Manchester, Ind., and graduated from the collegiate department of the latter school in 1892. In 1894 he entered Indiana University and graduated from the Department of Liberal Arts in 1896 with the degree of A.B. In this course Mathematics was his major subject. During the year 1898–99 he studied in both the University of Chicago and Indiana University, receiving the degree of Master of Arts from the latter institution at the close of the collegiate year, with Mechanics and Astronomy as major subject. His thesis was upon "The Path of a Particle Which is Subject to a Central Force Which Attracts Inversely as the Fourth Power of the Distance."

Mr. Hamilton, during the intermission of his collegiate and preparatory studies, taught in public school and college work. The positions held up to and including the year 1899–1900 are as follows:

In 1889–90, Teacher in Common School; 1891–92, Instructor in Latin and Algebra in North Manchester College; 1892–94, Principal of High School at Butler, Ind.; 1896–98, Superintendent of Schools at Hebron, Ind.; 1899–1900, Teacher of Mathematics in the California School of Mechanical Arts, San Francisco, Calif.

In 1900 Mr. Hamilton re-entered the University of Chicago, taking lectures in Astronomy and Mathematics. Including the time previously spent in this institution, he took courses as follows: under Dr. Laves: Theory of Orbits, Analytic Mechanics, Special Perturbations, Absolute Perturbations, Theory of Attractions of Heavenly Bodies, Spherical and Practical Astronomy; under Dr. Moulton: Method of Least Squares, Physical Astronomy, Problem of Three Bodies, Intro-

duction to Celestial Mechanics, Lunar Theory; under Mr. Lunn: The Problem of n Bodies; under Dr. Boyd: Theoretical Mechanics.

In Mathematics he took lectures as follows: under Professor Moore: Projective Geometry; under Professor Bolza: Definite Integrals, Theory of Functions of a Complex Variable and Elliptic Functions; under Professor Maschke: Fourier's Series and Elliptic Integrals, Theory of Functions of a Complex Variable and the Theory of Invariants; under Dr. Slaught: Definite Integrals and Differential Equations.